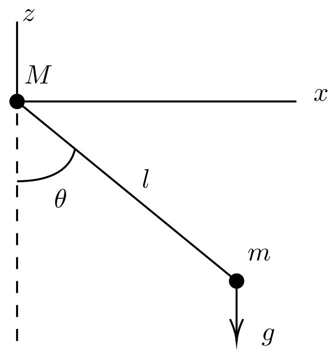
S.-T. Yau College Student Mathematics Contests 2026

Mathematical Physics

(6 problems)

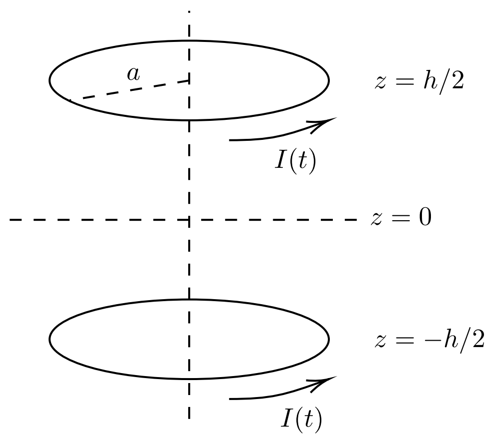
Problem 1. Consider a sliding pendulum, consisting of a mass M which can move without friction along a horizontal bar, and which is connected by a massless rod of length l to another mass m.



1. Derive the angular frequency of small oscillations in the $x-z$ plane.
2. Next consider the sliding conical (also called spherical) pendulum, for which the mass M can move without friction in the horizontal $x-y$ plane. Consider circular motion of both masses for a fixed, not necessarily small, angle θ. What is the angular frequency as a function of θ? Check your result for small θ.
3. Now consider the inverted sliding pendulum (which can only move in $x-z$ plane, i.e. back to the set-up of 1). As it starts falling from rest at $\theta = \pi$, there comes a point where the tension T in the rod becomes zero. Find the value of θ when this happens, assuming for simplicity that M is infinitely large.
4. Same set-up as 3. What is the tension at $\theta = \frac{\pi}{2}$ and $\theta = \pi$ when the inverted sliding pendulum is falling from rest at $\theta = \pi$ but this time M and m are arbitrary?

   (Hint: For an ellipse given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the curvature at the extremum with $x = a$ has a radius $\rho = \frac{b^2}{a}$, and at $y = b$ one has $\rho = \frac{a^2}{b}$.

Problem 2. Two identical circular coils of wire of radius a are separated by a length h. The coils each carry a slowly varying sinusoidal current $I(t) = I_0 \cos(\omega t)$. The axis of the coils is aligned with the z-axis and the geometry is centered at $z = 0$. (See below)



1. At lowest order in the frequency, a magnetostatic approximation is valid. Using this approximation, show that close to the axis, and near $z = 0$, the Taylor series for the axial and radial components of the slightly

off-axis magnetic field take the approximate form:

$$B_z \simeq B_0 + \frac{1}{2}\beta_2\left(z^2 - \frac{\rho^2}{2}\right) + \dots \tag{1}$$

$$B_\rho \simeq -\frac{1}{2}\beta_2 z\rho + \dots \tag{2}$$

where B_0 and β_2 are determined by the Taylor series of the magnetic field on the z-axis:

$$B_z(z) \simeq B_0 + \frac{1}{2}\beta_2 z^2 + \dots \tag{3}$$

Here $\rho = \sqrt{x^2 + y^2}$.

2. Using the magnetostatic approximation, determine the magnetic field in the z direction close to the axis of the solenoid, and near $z = 0$, to quadratic order in z and ρ. Describe the magnetic field when $h = a$.

3. Determine the electric field close to the axis of the solenoid at $z = 0$ to the lowest non-trivial order in the frequency and ρ.

4. Briefly answer the following:

   (a) For parts 2 and 3, give an estimate for the size of finite-frequency corrections.

   (b) For part 2, estimate at what large z the magnetostatic approximation breaks down when computing the magnetic field on the z-axis.

Problem 3. Consider a one-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential $U(x)$ composed of two static delta-functions

$$U(x) = \beta\left[\delta(x) + \delta(x - a)\right] \tag{4}$$

1. Can the particle tunnel through this barrier without reflection? Explain your answer.

2. If so, at what value(s) of the kinetic energy does this happen?

Now consider a three-dimensional non-relativistic particle of mass m and kinetic energy E scattering off the potential $U(\vec{r})$ composed of two static delta-functions

$$U(\vec{r}) = \beta_3(\delta^{(3)}(\vec{r} - \vec{r}_1) + \delta^{(3)}(\vec{r} - \vec{r}_2)). \tag{5}$$

Suppose that for each of the delta-functions in the above equation the S-wave scattering length $a < 0$.

3. Write explicitly the S-wave bound wave-function near each of the centers. Check that each does not support a bound state for $a < 0$.

4. Can the potential $U(\vec{r})$ with two centers support a bound state? If so, under what conditions? Explain your answer.

Hint: For a hard core potential of radius R, the S-wave scattering length a is defined as $d\log\chi(R)/dR = -1/a$, with $\chi(r)$ the S-wave reduced wave-function $\chi(r) \equiv r\psi(r)$. If $\psi(r)$ corresponds to a bound state, think about the constraints on the scattering length a.

Problem 4.

1. Write down the condition for thermodynamic equilibrium between the liquid and gas phases along a liquid-gas coexistence curve.

2. Using your solution to part 1 and taking into account the entropy change at a liquid-gas phase transition, derive the relation for the vapor pressure along a liquid-gas coexistence curve.

Problem 5. Consider the ϕ^4 model with a real scalar field $\phi(x)$ in $3+1$ dimensional Minkowski spacetime with metric $(-,+,+,+)$. Its Lagrangian density is

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}g\phi^4. \tag{6}$$

1. Write down the propagator and the interaction vertex for this model in momentum space.
2. Compute the one-loop correction to the 4−point function using dimensional regularization.
3. Determine the counterterm needed to cancel the divergence and write the renormalized coupling at 1−loop order (You may use minimal subtraction).
4. Write down the renormalization group flow equation for coupling constant.

You may find the following formula useful

$$(AB)^{-1} = \int_0^1 dx[xA + (1-x)B]^{-2}, \tag{7}$$

$$\int d^dk \frac{1}{(-k^2 - 2p\cdot k - M^2 + i\epsilon)^s} = (-1)^s i\pi^{d/2}\frac{\Gamma(s-d/2)}{\Gamma(s)}(-p^2 + M^2 - i\epsilon)^{d/2-s}, \tag{8}$$

where $\Gamma(z)$ is the Gamma function which has a simple pole at the origin.

$$\Gamma(z) = \frac{1}{z} - \gamma + \mathcal{O}(z), \tag{9}$$

where γ is the Euler constant.

Problem 6. The Klein disk model is an analytical description of non-Euclidean geometry in terms of coordinates (x^1, x^2) for points inside a unit disk. The distance $d(\vec{x}, \vec{y})$ between a point with coordinates $\vec{x} = (x^1, x^2)$ and another point with coordinates $\vec{y} = (y^1, y^2)$, both inside the unit disk, is defined to be

$$\cosh d(\vec{x}, \vec{y}) = \frac{1 - \vec{x}\cdot\vec{y}}{\sqrt{1 - \vec{x}\cdot\vec{x}}\sqrt{1 - \vec{y}\cdot\vec{y}}} \tag{10}$$

where $\vec{x}\cdot\vec{y} = x^1y^1 + x^2y^2$.

1. Compute the metric with components g_{11}, g_{12} and g_{22} from the formula for d by considering $\vec{y}$ near $\vec{x}$ (so $y^j = x^j + dx^j$, $j = 1, 2$).
2. Introduce the standard polar coordinates (r, θ) and write the line elements $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ in polar coordinates.
3. Determine the nonvanishing Christoffel symbols in polar coordinates.
4. Compute the component $R_{r\theta r\theta}$ of the Riemann tensor.
5. Show that for the Klein disk model, the Riemann curvature has the form

   $$R_{\mu\nu\rho\sigma} = K(g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}), \tag{11}$$

   and determine the value of K.
6. Compute the scalar curvature.